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On the stability of the inertial circulation in the 1$_{1/2}$-layer, quasi-geostrophic model

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Abstract. The effect of the time-dependent interface, separating an inertial quasi-geostrophic upper fluid layer from the quiescent abyss, on the non-linear stability of a steady circulation that takes place in this layer is explored. The analysis resorts to the method of Arnol'd's invariant resulting in a conditional stability criterion, which proves the stabilizing effect of the interface with respect to the single-layer case. The uniqueness of the stable basic flow field follows. Finally, non-linear and linear analyses are compared in the special case of a channeled flow with a fluctuating interface, the latter leading to an unconditional stability statement, whose meaning is clarified by resorting to the previously obtained nonlinear criterion.

PACS. 92.10.Fj Dynamics of the upper ocean – 47.20.Ky Nonlinearity (including bifurcation theory)

1 Introduction

1.1 The physical system

In the framework of the quasi-geostrophic dynamics, the potential vorticity conservation principle, valid for inertial flows evolving on the beta plane, is assumed as the governing equation also in the special case of a statically stable, two-layered unforced fluid. This kind of system, characterized by a rather simple vertical structure, is widely investigated in the literature and it is analyzed with full details for instance in Pedlosky ([1], Sect. 6.16). Here we review shortly the basic concepts. At the geostrophic level of approximation, the potential vorticity of the fluid columns within each layer is the sum of three terms of comparable magnitude: the relative vorticity of the geostrophic current, the deformation of the fluctuating interface between the layers (which provokes stretching and squeezing of the columns) and, finally, the contribution of the planetary vorticity (which is a linear function of the latitude of each column in the beta plane frame of reference). If we denote with ψ_1 and ψ_2 the perturbation pressure (non-dimensional quantities are hereafter understood) in the upper and lower layer respectively, the associated currents are $\mathbf{u}_1 = \hat{\mathbf{k}} \times \boldsymbol{\nabla}\psi_1$ and $\mathbf{u}_2 = \hat{\mathbf{k}} \times \boldsymbol{\nabla}\psi_2$ where $\hat{\mathbf{k}}$ is the unit vector normal to the beta plane and $\boldsymbol{\nabla}$ is the planar gradient operator $\boldsymbol{\nabla} = (\partial/\partial x,\ \partial/\partial y)$. Hence the first contributions to the potential vorticity, *i.e.* the relative vorticity in the upper and lower layer, are $\zeta_1 = \hat{\mathbf{k}} \cdot \boldsymbol{\nabla} \times \mathbf{u}_1 = \boldsymbol{\nabla}^2\psi_1$ and $\zeta_2 = \hat{\mathbf{k}} \cdot \boldsymbol{\nabla} \times \mathbf{u}_2 = \boldsymbol{\nabla}^2\psi_2$, respectively. The mechanism of stretching and squeezing of the fluid columns is described by the terms $-F_1(\psi_1 - \psi_2)$ and $-F_2(\psi_2 - \psi_1)$, which form the second contribution to potential vorticity. The parameters $F_i = f_0^2 L^2 (g D_i \Delta\rho/\rho)^{-1}$ $(i = 1, 2)$ play the role of coupling constants between the layers and depend on the Coriolis parameter f_0, the horizontal scale of the motion L, the relative density difference between the layers $\Delta\rho/\rho$ and the typical thickness of each layer D_i. Usually, the factor

$$R_i = \frac{1}{f_0}\left(g\frac{\Delta\rho}{\rho}D_i\right)^{\frac{1}{2}},$$

appearing in the definition of F_i, is called the Rossby deformation radius of the layer "i". The contribution of planetary vorticity to potential vorticity is given by the term βy, where β is the (non-dimensional) planetary vorticity gradient while y is the poleward Cartesian coordinate of the beta plane. On the whole, the potential vorticity P_i of each layer turns out to be

$$P_1 = \boldsymbol{\nabla}^2\psi_1 - F_1(\psi_1 - \psi_2) + \beta y$$
$$P_2 = \boldsymbol{\nabla}^2\psi_2 - F_2(\psi_2 - \psi_1) + \beta y$$

and the conservation equations in local form are

$$\frac{\partial P_i}{\partial t} + J(\psi_i, P_i) = 0 \quad (i = 1, 2).$$

In the last equation we have expressed the advective terms $\mathbf{u}_i \cdot \boldsymbol{\nabla}$ by using the known relations $\mathbf{u}_i = \hat{\mathbf{k}} \times \boldsymbol{\nabla}\psi_1$ and the Jacobian determinant

$$J(\psi_i, P_i) = \frac{\partial\psi_i}{\partial x}\frac{\partial P_i}{\partial y} - \frac{\partial\psi_i}{\partial y}\frac{\partial P_i}{\partial x}\,.$$

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Finally, by resorting to the explicit form of P_i and the identity $J(a,a)=0$, we have

$$\frac{\partial}{\partial t}(\nabla^2\psi_1 - F_1(\psi_1-\psi_2)) + J(\psi_1, \nabla^2\psi_1 + F_1\psi_2 + \beta y) = 0$$
$$\frac{\partial}{\partial t}(\nabla^2\psi_2 - F_2(\psi_2-\psi_1)) + J(\psi_2, \nabla^2\psi_2 + F_2\psi_1 + \beta y) = 0.$$

Can the lower layer represent a quiescent abyss? This situation would correspond to the choice $\psi_2 = 0$ everywhere, but, in such a case we would obtain, from the equation of the lower layer, $F_2 \partial\psi_1/\partial t = 0$. If we do not wish to constrain the upper layer to steadyness, the only possibility is $F_2 = 0$. Actually, equation $F_2 = 0$ means $f_0^2 L^2 (gD_2 \Delta\rho/\rho)^{-1} = 0$ and it can be satisfied only asymptotically for a very large value of the lower layer thickness D_2. The corresponding large value of the Rossby deformation radius R_2 means that the interface plays the role of a rigid lid for the lower layer. In the limit of infinite depth, the evolution of the fluid in the upper layer is given by the simplified equation $\partial(\nabla^2\psi_1 - F_1\psi_1)/\partial t + J(\psi_1, \nabla^2\psi_1 + \beta y) = 0$, which hereafter we write without the subscript 1:

$$\frac{\partial}{\partial t}(\nabla^2\psi - F\psi) + J(\psi, \nabla^2\psi + \beta y) = 0. \tag{1.1}$$

Equation (1.1) governs the so-called 11/2-layer model of inertial circulation, which is an intermediate case between the two-layer model and the fully barotropic model. The "infinite" thickness of the lower layer decouples the motion of the lighter fluid from that of the heavier, in spite of the presence of a fluctuating interface that separates the two fluids. Indeed, the interface plays a remarkable role also from the stability point of view, in the sense that it can introduce a stabilizing effect (far from being obvious) on the perturbed mean motion of the upper layer, as we will see in Section 3. Basically, we will take into account a square fluid domain, D, along which the no mass flux boundary condition is imposed

$$\psi = 0 \quad \forall (x,y) \in \mathsf{D} \quad \forall t \geq 0. \tag{1.2}$$

Conventionally, the stability of the perturbed solution will be explored from $t=0$ onwards. To this purpose, we define the basic state ψ_0 as the solution of the time independent problem constituted by the equation

$$J(\psi_0, \nabla^2\psi_0 + \beta y) = 0 \tag{1.3}$$

with the boundary condition

$$\psi_0 = 0 \quad \forall (x,y) \in \mathsf{D}, \tag{1.4}$$

which is the steady-state version of problem (1.1, 1.2). We introduce the potential vorticity Q of the basic state

$$Q \equiv \nabla^2\psi_0 - F\psi_0 + \beta y \tag{1.5}$$

and recall that (1.3) is satisfied if a functional relationships of the kind

$$\psi_0(x,y) = \Psi_0(Q(x,y)) \tag{1.6}$$

holds and Ψ_0 is a differentiable function of its argument. Each specific form of relationship (1.6) consistent with (1.4) singles out a basic state. Our main aim is to analyze, in terms of the functional dependence (1.6), how the non-linear stability of ψ_0 is affected by the term $-F\partial\psi/\partial t$ that involves the available potential energy of the disturbances superimposed to ψ_0 itself.

1.2 The mathematical method

We express every solution ψ of the time dependent problem (1.1, 1.2) as the superposition of a steady solution ψ_0 of the class (1.6) with a suitable time dependent disturbance ϕ

$$\psi = \psi_0 + \phi.$$

We briefly recall the concept of (nonlinear) stability. Once we have fixed a norm $N[\phi]$ and denoted by ϕ_i the disturbance evaluated in $t=0$, we say that ψ_0 is *stable* in the norm N if

$$\forall\varepsilon > 0 \;\; \exists\delta(\varepsilon) > 0\text{:} \;\; N[\phi_i] < \delta \Rightarrow N[\phi(t)] < \varepsilon \;\; \forall t \geq 0. \tag{1.7}$$

In plain words, the basic state ψ_0 is stable if the perturbed state ψ remains arbitrarily close to ψ_0 at every time provided that it is sufficiently close to ψ_0 at the initial time. Note, in particular, that (1.7) is verified if $N[\phi(t)] \leq N[\phi_i]$: in this case it is sufficient to take $\delta(\varepsilon) = \varepsilon$. (Hereafter, square brackets will be used mainly to denote functionals, *i.e.*, rules that associate numbers to functions.)

The method of Arnol'd's invariant [2–4] is a powerful tool to investigate the non-linear stability of planar, steady flows in inviscid fluids. It is based on a theorem about the finite-amplitude conservation of a functional, the invariant, that can be bounded from above and from below by linear combinations of squares of suitable perturbation norms $n_k^2[\phi]$ with $k = 0,1,2...$ In the lower bound, these norms are evaluated at a generic time t after the "initial" time $t=0$, while, in the upper bound, the same norms refer just to $t=0$. Then, if these linear combinations constitute the square of a norm, this norm "sandwiches" the invariant in t and in $t=0$; thus, the standard definition of stability is satisfied.

Formally, the situation is the following. Let $A[\psi_0,\phi]$ be the invariant (whose structure will be described in next section) and express its conservation through the equation

$$A[\psi_0, \phi(t)] = A[\psi_0, \phi_i]. \tag{1.8}$$

If we can write

$$\sum_k c_k n_k^2[\phi(t)] \leq A[\psi_0, \phi(t)] = A[\psi_0, \phi_i] \leq \sum_k \tilde{c}_k n_k^2[\phi_i], \tag{1.9}$$

where the summation will be specified in equation (3.7) and

$$0 \le \tilde{c}_k \le c_k \quad \forall k, \tag{1.10}$$

then the nonlinear stability of ψ_0 in the norms

$$N[\phi] \equiv \left\{ \sum_k c_k n_k^2[\phi] \right\}^{\frac{1}{2}} \quad \text{and} \quad \tilde{N}[\phi] \equiv \left\{ \sum_k \tilde{c}_k n_k^2[\phi] \right\}^{\frac{1}{2}} \tag{1.11}$$

follows immediately. In fact, according to (1.9),

$$N[\phi(t)] \le \tilde{N}[\phi_i] \tag{1.12}$$

while, from (1.10), we have

$$\tilde{N}[\phi(t)] \le N[\phi(t)] \quad \forall t \ge 0 \tag{1.13}$$

so (1.12, 1.13) imply both $N[\phi(t)] \le N[\phi_i]$ and $\tilde{N}[\phi(t)] \le \tilde{N}[\phi_i]$, in accordance with (1.7). Things are not so immediate if (1.10) does not hold. In fact, if a given coefficient of the linear combinations is negative, then the functional $N[\phi]$ is not convex, so it is not a norm nor can be used to analyze the stability of ψ_0. This situation happens whenever (1.6) yields a constraint on its derivative of the kind

$$-C \le \frac{\mathrm{d}\Psi_0}{\mathrm{d}Q} \le -c, \tag{1.14}$$

where C and c are two positive constants that depend on the form of (1.6) itself.

In this paper the method of Arnol'd's invariant to the steady solution ψ_0 of equation (1.1), under the assumption of the double constraint (1.14) is applied. The main result is a satisfactory definition of a perturbation norm suitable to prove the conditional stability of the system. Somewhat surprisingly, this approach is also able to explain why the interface has a stabilizing effect on the motion.

We take into account a rectangular fluid domain D included into the non-dimensional beta plane defined by

$$\mathsf{D} \equiv [x' \le x \le x' + \Delta x] \times [y' \le y \le y' + \Delta y] \tag{1.15}$$

and impose the no mass flux boundary condition (1.2) or (1.4) to the flow fields.

Finally, it is useful to recall two basic inequalities involving the integrated enstrophy as well as the kinetic and potential energy of the flow. They are (details can be found, for instance, in [5]):

$$B \int_{\mathsf{D}} \psi^2 \mathrm{d}x\mathrm{d}y \le \int_{\mathsf{D}} |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}y \tag{1.16}$$

and

$$B \int_{\mathsf{D}} |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}y \le \int_{\mathsf{D}} (\nabla^2 \psi)^2 \mathrm{d}x\mathrm{d}y. \tag{1.17}$$

The constant B appearing in (1.16, 1.17) depends, in general, on the shape and size of the fluid domain. For the domain (1.15) we have

$$B = \left(\frac{\pi}{\Delta x}\right)^2 + \left(\frac{\pi}{\Delta y}\right)^2 \cdot \tag{1.18}$$

2 Construction of Arnol'd's invariant

A short summary of the construction of Arnol'd's invariant, as reported in [4] but taking into account also the term $-F\partial\psi/\partial t$ of the governing equation (1.1) is outlined. The method is mainly based on the matching of two expressions of the kind

$$\frac{\mathrm{d}E[\phi]}{\mathrm{d}t} = \theta[\psi_0, \phi] \tag{2.1}$$

and

$$\frac{\mathrm{d}I[\psi_0, \phi]}{\mathrm{d}t} = -\theta[\psi_0, \phi] \tag{2.2}$$

that directly lead to the conservation statement

$$\frac{\mathrm{d}A[\psi_0, \phi]}{\mathrm{d}t} = 0 \tag{2.3}$$

where E, θ, I are functionals (to be determined) of their arguments, and $A[\psi_0, \phi] = E[\phi] + I[\psi_0, \phi]$ is just the invariant considered in the introduction.

Consider first (2.1) in which $E[\phi]$ means the integrated mechanical energy of the disturbance. Substitution of $\psi = \psi_0 + \phi$ into (1.1), recalling also (1.5), gives the equation

$$\frac{\partial}{\partial t}(\nabla^2 \phi - F\phi) + J(\psi_0, \nabla^2 \phi) + J(\phi, Q + F\psi_0 + \nabla^2 \phi) = 0. \tag{2.4}$$

The boundary condition of ϕ easily comes from (1.2, 1.4):

$$\phi = 0 \quad \forall (x, y) \in \partial\mathsf{D}. \tag{2.5}$$

Multiplication of (2.4) by ϕ and the subsequent integration on D with the aid of (2.5) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathsf{D}} \left(|\nabla \phi|^2 + F\phi^2 \right) \mathrm{d}x\mathrm{d}y = \int_{\mathsf{D}} \nabla^2 \phi J(\phi, \psi_0) \mathrm{d}x\mathrm{d}y. \tag{2.6}$$

Equation (2.6) has the same form as (2.1) if we designate $E[\phi]$ and $\theta[\psi_0, \phi]$ through

$$E[\phi] \equiv \frac{1}{2} \int_{\mathsf{D}} \left(|\nabla \phi|^2 + F\phi^2 \right) \mathrm{d}x\mathrm{d}y, \tag{2.7}$$
$$\theta[\psi_0, \phi] \equiv \int_{\mathsf{D}} \nabla^2 \phi J(\phi, \psi_0) \mathrm{d}x\mathrm{d}y.$$

Consider now (2.2). Define the integral

$$H(Q, q) = \int_0^q [\Psi_0(Q + \xi) - \Psi_0(Q)] \mathrm{d}\xi \tag{2.8}$$

where, for the moment, $q = q(x, y, t)$ is left unspecified. From (2.8) we evaluate the time derivative to obtain

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial H}{\partial Q}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\partial H}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} \\ &= \left[\Psi_0(Q + q) - \Psi_0(Q) - q\frac{\partial \psi_0}{\partial Q} \right] \frac{\mathrm{d}Q}{\mathrm{d}t} \\ &\quad + [\Psi_0(Q + q) - \Psi_0(Q)] \frac{\mathrm{d}q}{\mathrm{d}t} \cdot \end{aligned} \tag{2.9}$$

Now, if $P = \boldsymbol{\nabla}^2\psi - F\psi + \beta y$ is the potential vorticity of the perturbed state and we set

$$q \equiv \boldsymbol{\nabla}^2\phi - F\phi, \tag{2.10}$$

then, recalling (1.5), we have $P = Q + q$ and, because of (1.1), $\mathrm{d}Q/\mathrm{d}t = -\mathrm{d}q/\mathrm{d}t$. Moreover,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + J(\psi_0 + \phi, Q) = J(\phi, Q)$$

where the steadiness of Q has been used in the last step. Therefore, (2.9) becomes

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -q\frac{\partial\psi_0}{\partial Q}J(\phi, Q) = -(\boldsymbol{\nabla}^2\phi - F\phi)J(\phi, \psi_0). \tag{2.11}$$

Finally, integration of (2.11) on D with the aid of (2.5) yields, recalling also (2.8),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{D}}\left\{\int_0^q [\Psi_0(Q+\xi) - \Psi_0(Q)]\mathrm{d}\xi\right\}\mathrm{d}x\mathrm{d}y = -\int_{\mathsf{D}} \boldsymbol{\nabla}^2\phi J(\phi, \Psi_0)\mathrm{d}x\mathrm{d}y. \tag{2.12}$$

Comparison of (2.12) with (2.2) shows that

$$I[\psi_0, \phi] \equiv \int_{\mathsf{D}}\left\{\int_0^q [\Psi_0(Q+\xi) - \Psi_0(Q)]\mathrm{d}\xi\right\}\mathrm{d}x\mathrm{d}y. \tag{2.13}$$

On the whole, from (2.1, 2.2, 2.3, 2.7, 2.13), Arnol'd's invariant turns out to be

$$A[\psi_0, \phi] = \int_{\mathsf{D}}\left\{\frac{1}{2}\left(|\boldsymbol{\nabla}\phi|^2 + F\phi^2\right) + \int_0^q (\Psi_0(Q+\xi) - \Psi_0(Q)]\mathrm{d}\xi\right\}\mathrm{d}x\mathrm{d}y. \tag{2.14}$$

3 Determination of the norm associated to Arnol'd's invariant and the effect of a moving interface on stability

In order to take into account explicitly the double inequality (1.14), it is useful to apply Lagrange theorem to the last integrand of (2.14)

$$\Psi_0(Q+\xi) - \Psi_0(Q) = \left(\frac{\mathrm{d}\Psi_0}{\mathrm{d}Q}\right)_{Q=\bar{Q}}\xi, \tag{3.1}$$

where $\bar{Q}$ is a suitable value of the potential vorticity of the basic state. Applying (1.14) to (3.1), we have

$$\frac{c}{2}q^2 \le -\int_0^q [\Psi_0(Q+\xi) - \Psi_0(Q)]\,\mathrm{d}\xi \le \frac{C}{2}q^2 \tag{3.2}$$

and, on the whole, we can write

$$\int_{\mathsf{D}}\left\{cq^2(t) - |\boldsymbol{\nabla}\phi(t)|^2 - F\phi^2(t)\right\}\mathrm{d}x\mathrm{d}y \le -2A[\psi_0, \phi(t)] = -2A[\psi_0, \phi_i] \le \int_{\mathsf{D}}\{Cq_i^2 - |\boldsymbol{\nabla}\phi_i|^2 - F\phi_i^2\}\mathrm{d}x\mathrm{d}y \le \int_{\mathsf{D}} Cq_i^2\mathrm{d}x\mathrm{d}y \tag{3.3}$$

(the factor -2 is insignificant since we can redefine the invariant $A \to -2A$).

At this point it is useful to define the norms:

$$n_1[\phi] \equiv \left\{\int_{\mathsf{D}} q^2\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} \tag{3.4}$$

and

$$n_0[\phi] \equiv \left\{\int_{\mathsf{D}}\left(|\boldsymbol{\nabla}\phi|^2 + F\phi^2\right)\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} \tag{3.5}$$

so that (3.3) can be rewritten as

$$cn_1^2[\phi(t)] - n_0^2[\phi(t)] \le -2A[\psi_0, \phi(t)] = -2A[\psi_0, \phi_i] \le Cn_1^2[\phi_i]. \tag{3.6}$$

With reference to (1.9) we identify

$$c_0 = -1,\; c_1 = c,\; \tilde{c}_0 = 0,\; \tilde{c}_1 = C,\; \tilde{c}_k = c_k = 0 \quad \forall k > 1 \tag{3.7}$$

where c and C are the constants appearing in (1.14).

In the introduction, we have anticipated that for proper values of c we have $cn_1^2[\phi(t)] - n_0^2[\phi(t)] > 0\ \forall\phi$, in general, however one of the defining axioms of a norm, namely the triangular inequality, does not hold. In spite of this, a possible way out is outlined in what follows.

If an inequality of the kind

$$n_0^2[\phi] \le \lambda n_1^2[\phi] \tag{3.8}$$

does exist, and

$$0 < \lambda < c \tag{3.9}$$

then (3.6) implies the inequalities:

$$n_1^2[\phi(t)] \le \frac{1}{c-\lambda}Cn_1^2[\phi_i] \tag{3.10}$$

and

$$n_0^2[\phi(t)] \le \frac{\lambda}{c-\lambda}Cn_1^2[\phi_i]. \tag{3.11}$$

Addition of (3.11) with (3.10) multiplied by λ can be bounded from above according to the inequality

$$n_0^2[\phi(t)] + \lambda n_1^2[\phi(t)] \le \frac{2C}{c-\lambda}\left\{n_0^2[\phi_i] + \lambda n_1^2[\phi_i]\right\}\cdot \tag{3.12}$$

If we define the norm

$$n[\phi] \equiv \{n_0^2[\phi] + \lambda n_1^2[\phi]\}^{\frac{1}{2}}, \tag{3.13}$$

then inequality (3.12) implies the stability of the basic state ψ_0 *in the norm* (3.13); in fact we have

$$n[\phi(t)] \le \left\{\frac{2C}{c-\lambda}\right\}^{\frac{1}{2}} n[\phi_i]. \tag{3.14}$$

Inequality (3.14) is especially useful if we are able to find the least upper bound of the quantity λ appearing in (3.8). Recalling (2.10, 3.5, 3.6, 3.8), the problem involving λ takes the form

$$R[\phi] \equiv \frac{\int_{\mathsf{D}} \left(|\boldsymbol{\nabla}\phi|^2 + F\phi^2\right) \mathrm{d}x\mathrm{d}y}{\int_{\mathsf{D}} [(\boldsymbol{\nabla}^2\phi)^2 + 2F|\boldsymbol{\nabla}\phi|^2 + F^2\phi^2]\,\mathrm{d}x\mathrm{d}y} \le \lambda. \tag{3.15}$$

To develop (3.15) we take a K such that $0 \le K \le 1$ and bound from below the integral

$$\int_{\mathsf{D}} (\boldsymbol{\nabla}^2\phi)^2 \mathrm{d}x\mathrm{d}y \equiv K \int_{\mathsf{D}} (\boldsymbol{\nabla}^2\phi)^2 \mathrm{d}x\mathrm{d}y + (1-K) \int_{\mathsf{D}} (\boldsymbol{\nabla}^2\phi)^2 \mathrm{d}x\mathrm{d}y$$

in terms of $\int_{\mathsf{D}} |\boldsymbol{\nabla}\phi|^2 \mathrm{d}x\mathrm{d}y$ and $\int_{\mathsf{D}} \phi^2 \mathrm{d}x\mathrm{d}y$ by resorting to (1.16) and (1.17). We obtain

$$\int_{\mathsf{D}} (\boldsymbol{\nabla}^2\phi)^2 \mathrm{d}x\mathrm{d}y \ge KB \int_{\mathsf{D}} |\boldsymbol{\nabla}\phi|^2 \mathrm{d}x\mathrm{d}y + (1-K)B^2 \int_{\mathsf{D}} \phi^2 \mathrm{d}x\mathrm{d}y. \tag{3.16}$$

Further, using (3.16) in (3.15), we get

$$R[\phi] \le \frac{\int_{\mathsf{D}} \left(|\boldsymbol{\nabla}\phi|^2 + F\phi^2\right) \mathrm{d}x\mathrm{d}y}{(KB+2F)\int_{\mathsf{D}} |\boldsymbol{\nabla}\phi|^2 \mathrm{d}x\mathrm{d}y + [(1-K)B^2+F^2]\int_{\mathsf{D}} \phi^2 \mathrm{d}x\mathrm{d}y} \cdot \tag{3.17}$$

If

$$(1-K)B^2 + F^2 = F(KB + 2F), \tag{3.18}$$

then (3.17) immediately yields

$$R[\phi] \le \frac{1}{KB + 2F} \tag{3.19}$$

and an upper bound of $R[\phi]$ is found. Equation (3.18) gives

$$K = \frac{B - F}{B} \cdot \tag{3.20}$$

Solution (3.20) is constrained by the double inequality

$$0 \le \frac{B-F}{B} \le 1$$

that indeed demands only $B \ge F$. This last inequality is easily met, for instance, by choosing $\Delta x = \Delta y = 1$ in (1.15) and $F = 1$ in (1.1), in accordance with the non-dimensionality of the problem under investigation. In this case $B \ge F$ means (see Eq. (1.18)) $2\pi^2 \ge 1$, which is trivially true. Substitution of (3.20) into (3.19) yields $R[\phi] \le 1/(B+F)$ and, according to (3.9), the conditional stability of ψ_0 in the norm (3.13) is ensured if $c > 1/(B+F)$, that is to say, with reference to (1.14), if

$$-C \le \frac{\mathrm{d}\psi_0}{\mathrm{d}Q} \le -c < -\frac{1}{B+F} \cdot \tag{3.21}$$

Inequality (3.21) represents our basic result. The remarkable difference with respect to the barotropic case ($F = 0$) is the presence of a fluctuating interface ($F > 0$) between the moving layer and the quiescent abyss that stabilizes the flow, in the sense that it enlarges towards zero the interval of values of c corresponding to stable solutions. Our result is consistent with that of [6], which was obtained within a rather different context.

Finally, we prove that

$$\lambda = \frac{1}{B+F} \tag{3.22}$$

is the best estimate of the upper bound (3.15). In fact, if $\lambda_* > 1/(B+F)$ were the best estimate, then the equation

$$n_0^2[\tilde{\phi}] = \left(\frac{1}{B+F} + a\right) n_1^2[\tilde{\phi}] \tag{3.23}$$

would be satisfied by some $\tilde{\phi}$, where $1/(B+F) + a < \lambda_*$ and $a > 0$. Equation (3.23) is equivalent to

$$(B+F)n_0^2[\tilde{\phi}] = (1 + a(B+F))n_1^2[\tilde{\phi}] \tag{3.24}$$

but we show that if $a > 0$, the lhs of (3.24) is necessarily lesser than the rhs of the same equation. In fact, the lhs is bounded from above according to the inequality

$$(B+F)n_0^2[\phi] < (1 + a(B+F))(B+F)n_0^2[\phi]. \tag{3.25}$$

On the other hand, we know from equations (3.8, 3.22) that $n_1^2[\phi] \ge (B+F)n_0^2[\phi]$, so the rhs of (3.24) is bounded from below as follows:

$$(1 + a(B+F))n_1^2[\tilde{\phi}] \ge (1 + a(B+F))(B+F)n_0^2[\tilde{\phi}]. \tag{3.26}$$

Since equations (3.25, 3.26) are not consistent with (3.24) for $a > 0$, we conclude that (3.22) is the best estimate.

4 Uniqueness of the stable basic state

In this section we show that, if $c > 1/(B+F)$ and $Q(\psi_0)$ is single-valued, then ψ_0 is the *unique* solution of the steady problem that defines it. We recall that the problem for ψ_0 is

$$\boldsymbol{\nabla}^2\psi_0 - F\psi_0 + \beta y = Q(\psi_0) \tag{4.1}$$

$$\psi_0 = 0 \ \ \forall (x,y) \in \partial\mathsf{D}. \tag{4.2}$$

Should we assume that problem (4.1, 4.2) has two solutions, say for example ψ_0^{I} and ψ_0^{II}, if we set $\delta \equiv \psi_0^{\mathrm{I}} - \psi_0^{\mathrm{II}}$ and subtract from (4.1) evaluated for ψ_0^{I} the same equation evaluated for $\psi_0^{\rm II}$, we obtain $\nabla^2\delta - F\delta = Q(\psi_0^{\rm I}) - Q(\psi_0^{\rm II})$. Hence, by resorting to the Lagrange theorem, we have

$$\nabla^2\delta - F\delta = \left[\frac{\mathrm{d}Q}{\mathrm{d}\psi_0}\right]_{\psi_0=\bar\psi_0}\delta. \tag{4.3}$$

Multiplication of (4.3) by δ and the subsequent integration on D with the aid of (4.2), which implies $\delta = 0\ \forall(x, y) \in \partial\mathsf{D}$, yields

$$-\int_{\mathsf{D}}|\nabla\delta|^2\mathrm{d}x\mathrm{d}y = \int_{\mathsf{D}}\left\{F + \left[\frac{\mathrm{d}Q}{\mathrm{d}\psi_0}\right]_{\psi_0=\bar\psi_0}\right\}\delta^2\mathrm{d}x\mathrm{d}y. \tag{4.4}$$

As $Q(\psi_0)$ is single-valued, inequality (1.14) implies $-1/c \le \mathrm{d}Q/\mathrm{d}\psi_0 \le -1/C$; therefore, using also (1.16), the rhs of (4.4) can be estimated as follows

$$\left(F - \frac{1}{c}\right)\int_{\mathsf{D}}\delta^2\mathrm{d}x\mathrm{d}y \le \int_{\mathsf{D}}\left\{F + \left[\frac{\mathrm{d}Q}{\mathrm{d}\psi_0}\right]_{\psi_0=\bar\psi_0}\right\}\delta^2\mathrm{d}x\mathrm{d}y \le -B\int_{\mathsf{D}}\delta^2\mathrm{d}x\mathrm{d}y$$

and hence

$$\left(B + F - \frac{1}{c}\right)\int_{\mathsf{D}}\delta^2\mathrm{d}x\mathrm{d}y \le 0. \tag{4.5}$$

As $c > 1/(B+F)$, inequality (4.5) can be satisfied only if $\delta = 0$, which implies $\psi_0^{\rm I} = \psi_0^{\rm II}$, *i.e.* the uniqueness of ψ_0.

5 Comparison between linear and non-linear stability of a channeled flow

It is interesting to compare the results of the stability analysis of a given basic flow both from the nonlinear and the linear points of view. To this purpose, we analyze the stability of a steady zonal flow of the kind

$$\psi_0 = \psi_0(y) \tag{5.1}$$

in the unbounded domain

$$\mathsf{D} =]-\infty < x < +\infty[\times[y' \le y \le y' + \Delta y], \tag{5.2}$$

under the no mass flux boundary condition

$$\psi_0(y') = \psi_0(y' + \Delta y) = 0. \tag{5.3}$$

We stress that criterion (3.21) can be easily modified within the non-linear framework under the hypotheses (5.1, 5.2), as follows. First of all, the potential vorticity of the basic state is

$$Q = \frac{\mathrm{d}^2\psi_0}{\mathrm{d}y^2} - F\psi_0 + \beta y. \tag{5.4}$$

Moreover, to ensure convergent integrals, we restrict the admissible perturbations to those periodic in longitude; that is, we assume $\phi(x, y) = \phi(x + \Lambda, y)$ and evaluate the norm squares appearing in Sections 2 and 3, *i.e.* $\int_{\mathsf{D}}\phi^2\mathrm{d}x\mathrm{d}y$, $\int_{\mathsf{D}}|\nabla\phi|^2\mathrm{d}x\mathrm{d}y$ and $\int_{\mathsf{D}}(\nabla^2\phi)^2\mathrm{d}x\mathrm{d}y$ in $\mathsf{D} \equiv \mathsf{D}_\Lambda$, where

$$\mathsf{D}_\Lambda = [x' \le x \le x' + \Lambda] \times [y' \le y \le y' + \Delta y].$$

Generally, $\phi(x', y)$ and $\phi(x'+\Lambda, y)$ are not zero, so the constant B appearing in (1.13, 1.14) is different from (1.15), and in the present section it takes the value

$$B = \left(\frac{\pi}{\Delta y}\right)^2. \tag{5.5}$$

On the whole, the form of criterion (3.21) still holds, but with equations (5.4, 5.5) instead of equations (1.5, 1.18) respectively.

In the linear framework, the term $J(\phi, \nabla^2\phi)$ is neglected with respect to the other terms of (2.4), which can be re-written as

$$\frac{\partial}{\partial t}(\nabla^2\phi - F\phi) + u_0\frac{\partial}{\partial x}\nabla^2\phi + \left(\beta - \frac{\mathrm{d}^2u_0}{\mathrm{d}y^2}\right)\frac{\partial\phi}{\partial x} = 0 \tag{5.6}$$

where we have introduced the zonal current $u_0 = -\mathrm{d}\psi_0/\mathrm{d}y$ of the basic state (5.1). According to the standard theory, we put

$$\phi = A(y)\exp[\mathrm{i}k(x - ct)] \tag{5.7}$$

where the complex amplitude $A(y)$ satisfies (5.3), *i.e.*

$$A(y') = A(y' + \Delta y) = 0 \tag{5.8}$$

while $c = \mathrm{Re}(c) + \mathrm{i}\,\mathrm{Im}(c)$ is the along-channel complex propagation velocity of the perturbation. Substitution of (5.7) into (5.6) yields

$$\frac{\mathrm{d}^2A}{\mathrm{d}y^2} - k^2A + \left(cF + \beta - \frac{\mathrm{d}^2u_0}{\mathrm{d}y^2}\right)(u_0 - c)^{-1}A = 0 \tag{5.9}$$

Integration of (5.9) multiplied by A^* with the aid of (5.8) gives

$$-\int_{y'}^{y'+\Delta y}\left(\left|\frac{\mathrm{d}A}{\mathrm{d}y}\right|^2 + k^2|A|^2\right)\mathrm{d}y + \int_{y'}^{y'+\Delta y}\left(cF+\beta-\frac{\mathrm{d}^2u_0}{\mathrm{d}y^2}\right)(u_0-c)^{-1}|A|^2\mathrm{d}y=0 \tag{5.10}$$

and the vanishing of the imaginary part of (5.10) is expressed by the equation

$$\mathrm{Im}(c)\int_{y'}^{y'+\Delta y}\left(u_0F + \beta - \frac{\mathrm{d}^2u_0}{\mathrm{d}y^2}\right)\left|\frac{A}{u_0 - c}\right|^2\mathrm{d}y = 0. \tag{5.11}$$

From (5.11) we see that if $u_0F + \beta - \mathrm{d}^2u_0/\mathrm{d}y^2$ has a constant sign $\forall y$ in the interval $]y', y' + \Delta y[$, then $\mathrm{Im}(c) = 0$

and the basic state ψ_0 is linearly stable. If $F = 0$, this statement is known as Kuo criterion ([7], reported, for instance, in [8], Sect. 7-2). Unlike in [7], here we take $F > 0$. If we define

$$m \equiv \underset{[y',y'+\Delta y]}{\text{Min}} \left(u_0 F + \beta - \frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} \right)$$

and

$$M \equiv \underset{[y',y'+\Delta y]}{\text{Max}} \left(u_0 F + \beta - \frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} \right),$$

we can trivially write

$$m \leq u_0 F + \beta - \frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} \leq M. \tag{5.12}$$

At this point, we resort to the fact that the stability property of a zonal flow does not change under a Galilean transform of the reference system, of the kind

$$\begin{cases} x \to x = \tilde{u}t \\ y \to y \end{cases} \tag{5.13}$$

where $\tilde{u}$ is any constant velocity in the zonal direction. With respect to the new frame (5.13), the zonal velocity is $u_0 + \tilde{u}$, while $\mathrm{d}^2 u_0/\mathrm{d}y^2$ is left unchanged by (5.13) and equation (5.12) takes the form

$$m + \tilde{u}F \leq u_0 F + \tilde{u}F + \beta - \frac{\mathrm{d}^2 u_0}{\mathrm{d}y^2} \leq M + \tilde{u}F. \tag{5.14}$$

Due to the arbitrariness of $\tilde{u}$, we can fix $\tilde{u}$ such that $m + \tilde{u}F > 0$ or $M + \tilde{u}F < 0$, *i.e.*

$$\tilde{u} > -\frac{m}{F} \text{ or } \tilde{u} < -\frac{M}{F} \cdot \tag{5.15}$$

If one of (5.15) holds, then $u_0 F + \tilde{u}F + \beta - \mathrm{d}^2 u_0/\mathrm{d}y^2$ has a constant sign inside $]y', y' + \Delta y[$ because of (5.14). Hence, the linear stability of ψ_0 follows in the reference frame (5.13) and hence in all the frames. In other words, every basic state (5.1) satisfying (5.3) turns out to be linearly stable.

6 Conclusions

In the above section two different conclusions depending on the non-linear or linear approach to the stability problem of the same basic state are discussed. Unconditional linear stability would seem to bring about a more general result, but it is strongly restricted by the linearity assumption itself. In fact, for the linear analysis only short time intervals dealing with small amplitude disturbances can be taken into account, omitting consideration of slowly growing perturbations, as is the case of resonant triads of Rossby waves ([1], Sect. 3.26). According to assumption (5.7), the linear analysis allows unstable states to grow only exponentially, so a possible slowly growing perturbation is (incorrectly) explained as the behaviour of a stable perturbed flow. On the contrary, the non-linear approach is based on the conservation principle (1.5) and therefore it renounces all hypotheses on the details of the time evolution of the system, thus always yielding well-grounded information.

Finally, we take the opportunity to point out the link between the definition (1.7) of nonlinear stability and the approach of the linear stability theory. The latter theory uses (implicitly) the norm defined by

$$N[\phi] = \int_{\mathsf{D}} \phi\phi^* \mathrm{d}x\mathrm{d}y,$$

so position (5.7) yields

$$N[\phi(t)] = \int_{\mathsf{D}} |A|^2 \exp[-2kI_m(c)t] \mathrm{d}x\mathrm{d}y \tag{6.1}$$

and hence

$$N[\phi_i] = \int_{\mathsf{D}} |A|^2 \mathrm{d}x\mathrm{d}y. \tag{6.2}$$

Now, as the system under investigation is inviscid, if $\mathrm{Re}(c) + \mathrm{i}\,\mathrm{Im}(c)$ is an eigenvalue of (5.9), also $\mathrm{Re}(c) - \mathrm{i}\,\mathrm{Im}(c)$ is an eigenvalue of the same equation. Therefore, the implication

$$\int_{\mathsf{D}} |A|^2 \mathrm{d}x\mathrm{d}y < \delta(\varepsilon) \Rightarrow \int_{\mathsf{D}} |A|^2 \exp[-2kI_m(c)t] \mathrm{d}x\mathrm{d}y < \varepsilon \quad \forall t \geq 0$$

coming from (1.7) with the use of equations (6.1, 6.2) is not met *for all* $\mathrm{Im}(c)$, unless $\mathrm{Im}(c) = 0$. In other words, stability holds only if all the eigenvalues of (5.9) are real. At this point, the nonlinear analysis should also have clarified the rationale behind the procedure used after equation (5.11) to show the linear stability of the Kuo flow.

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